

BRAIDED CLIFFORD ALGEBRAS AS BRAIDED QUANTUM GROUPS

MILCO ĐURĐEVIĆ

ABSTRACT. The paper deals with braided Clifford algebras, understood as Chevalley-Kähler deformations of braided exterior algebras. It is shown that Clifford algebras based on involutive braids can be naturally endowed with a braided quantum group structure. Basic group entities are constructed explicitly.

1. INTRODUCTION

The aim of this letter is to present a general construction of examples of braided quantum groups, based on Clifford algebras associated to involutive braids. From the noncommutative-geometric [1] point of view, these structures include completely pointless objects, and overcome in such a way an inherent geometrical inhomogeneity of standard quantum groups (caused by the presence of classical points). The construction is based on a general theory of braided Clifford algebras [2], interpreted as Chevalley-Kähler deformations of the corresponding braided exterior algebras. It will be shown that if the braid operator is involutive the corresponding Clifford algebras can be naturally equipped with a coalgebra structure and the antipode map, so that the whole structure becomes a braided quantum group [3].

The paper is organized as follows. In the next two sections basic properties of braided quantum groups and braided Clifford algebras are collected. The main construction of the group structure on braided Clifford algebras is given in Section 4. The construction is presented in two conceptually different ways. Finally, in the last section some concluding remarks are made.

2. BRAIDED EXTERIOR AND CLIFFORD ALGEBRAS

Let W be a (complex) finite-dimensional vector space, and let ψ be an arbitrary linear automorphism of $W \otimes W$ satisfying the braid equation

$$(\psi \otimes \text{id})(\text{id} \otimes \psi)(\psi \otimes \text{id}) = (\text{id} \otimes \psi)(\psi \otimes \text{id})(\text{id} \otimes \psi).$$

By definition [4], the corresponding *exterior algebra* W^\wedge is the factoralgebra of the tensor algebra W^\otimes relative to the ideal $\ker(A) \subseteq W^\otimes$. Here $A: W^\otimes \rightarrow W^\otimes$ is the corresponding total antisymetrizer map. Its components $A_n: W^{\otimes n} \rightarrow W^{\otimes n}$ are given by

$$A_n = \sum_{\pi \in S_n} (-1)^\pi \psi_\pi$$

where $\psi_\pi: W^{\otimes n} \rightarrow W^{\otimes n}$ are maps obtained by replacing transpositions figuring in a minimal decomposition of π by the corresponding ψ -twists. The following factorizations hold

$$(1) \quad A_{n+k} = (A_n \otimes A_k)A_{nk} \quad A_{n+k} = B_{nk}(A_n \otimes A_k)$$

where

$$A_{nk} = \sum_{\pi \in S_{nk}} (-1)^\pi \psi_{\pi^{-1}} \quad B_{nk} = \sum_{\pi \in S_{nk}} (-1)^\pi \psi_\pi$$

and $S_{nk} \subseteq S_{n+k}$ is the set of permutations preserving the order of sets $\{1, \dots, n\}$ and $\{n+1, \dots, n+k\}$.

The algebra W^\wedge can be naturally realized as a subspace $\text{im}(A) \subseteq W^\otimes$. This realization is given by

$$\left[\zeta + \ker(A) \right] \leftrightarrow A(\zeta).$$

In terms of the above identification,

$$\zeta \wedge \xi = B_{nk}(\zeta \otimes \xi)$$

for each $\zeta \in W^{\wedge n}$ and $\xi \in W^{\wedge k}$.

The map ψ can be naturally extended to a braiding on W^\wedge (which will be denoted by the same symbol) such that

$$(2) \quad \psi(\text{id} \otimes m) = (m \otimes \text{id})(\text{id} \otimes \psi)(\psi \otimes \text{id})$$

$$(3) \quad \psi(m \otimes \text{id}) = (\text{id} \otimes m)(\psi \otimes \text{id})(\text{id} \otimes \psi),$$

where m is the corresponding product map. By construction,

$$\psi(W^{\wedge i} \otimes W^{\wedge j}) = W^{\wedge j} \otimes W^{\wedge i}$$

for each $i, j \geq 0$.

Let us assume that W is endowed with a “quadratic form” F , which will be understood as a linear map $F: W \otimes W \rightarrow \mathbb{C}$ (or equivalently, as a bilinear map $F: W \times W \rightarrow \mathbb{C}$). Further, let us assume that F and ψ are mutually related in the following way

$$(4) \quad (F \otimes \text{id})(\text{id} \otimes \psi) = (\text{id} \otimes F)(\psi \otimes \text{id})$$

Let $\iota^F: W \times W^\wedge \rightarrow W^\wedge$ be a bilinear map specified by

$$\begin{aligned} \iota_x^F(\zeta) &= F(x, \zeta) \\ \iota_x^F(\vartheta \wedge \eta) &= \iota_x^F(\vartheta) \wedge \eta + (-1)^{\partial \vartheta} \sum_k \vartheta_k \wedge \iota_{x_k}^F(\eta) \end{aligned}$$

where $\sum_k \vartheta_k \otimes x_k = \psi(x \otimes \vartheta)$ and $\zeta \in W$. The second condition is a braided variant of the Leibniz rule. The map ι^F is uniquely (and consistently) determined by the above conditions. We have

$$\iota_x^F(1) = 0$$

for each $x \in W$.

The introduced contraction operator can be trivially extended to the map of the form $\iota^F: W^\otimes \times W^\wedge \rightarrow W^\wedge$ by requiring

$$\iota_{u \otimes v}^F = \iota_u^F \iota_v^F$$

for each $u, v \in W^\otimes$. It turns out that if $u \in \ker(A)$ then $\iota_u^F = 0$.

Therefore, it is possible to pass from W^\otimes to W^\wedge in the first argument of ι^F . In such a way we obtain a contraction map $\iota^F: W^\wedge \times W^\wedge \rightarrow W^\wedge$.

Let us define “relative” contraction operators $\langle \rangle_k: W^\wedge \times W^\wedge \rightarrow W^\wedge$ as bilinear maps of the form

$$\langle \vartheta, \eta \rangle_k = \sum_{\alpha\beta} \alpha \wedge \iota_\beta^F(\eta)$$

where

$$\sum_{\alpha\beta} \alpha \wedge \beta = A_{lk}(\vartheta^*)$$

while $\alpha \in W^{\wedge l}$, $\beta \in W^{\wedge k}$ and $\vartheta \in W^{\wedge n}$, with $n = k + l$. Finally, $\vartheta^* \in W^{\otimes n}$ is such that $[\vartheta^*]^\wedge = \vartheta$. Consistency of this definition follows from the first decomposition in (1).

The formula

$$(5) \quad \vartheta \circ \eta = \vartheta \wedge \eta + \sum_{k \geq 1} \langle \vartheta, \eta \rangle_k$$

defines a new F -dependent product in the space W^\wedge . The associativity of this product follows from (4). In particular,

$$x \circ \vartheta = x \wedge \vartheta + \iota_x^F(\vartheta)$$

for $x \in W$. This is a counterpart of classical Chevalley’s formula.

Endowed with \circ , the space W^\wedge becomes a unital associative algebra, with the unity $1 \in W^\wedge$.

By definition, the algebra $\mathfrak{cl}(W, \psi, F) = (W^\wedge, \circ)$ is called *the braided Clifford algebra* (associated to $\{\psi, F\}$).

The constructed algebra can be understood as a deformation of the exterior algebra W^\wedge . Furthermore, the graded algebra associated to the filtered $\mathfrak{cl}(W, \psi, F)$ naturally coincides with the braided exterior algebra W^\wedge .

The construction of braided Clifford algebras can be performed in a conceptually different way [2], introducing a Clifford product in the tensor algebra W^\otimes . Let us consider a map $\lambda_F: W^\otimes \rightarrow W^\otimes$ specified by

$$\lambda_F(1) = 1 \quad \lambda_F(x \otimes \vartheta) = x \otimes \lambda_F(\vartheta) + \iota_x^F \lambda_F(\vartheta)$$

where $x \in W$ and $\vartheta \in W^\otimes$. In the above formula ι^F is considered as a contraction acting in W^\otimes . The map λ_F is bijective. Let \circ be a new product in W^\otimes , given by

$$\vartheta \circ \eta = \lambda_F(\lambda_F^{-1}(\vartheta) \otimes \lambda_F^{-1}(\eta)).$$

By construction the space $\ker(A)$ is a left ideal in W^\otimes , relative to this new product. Condition (4) implies that $\ker(A)$ is also a right \circ -ideal.

Let $J_F \subseteq W^\otimes$ be the ideal corresponding to the constructed braided Clifford algebra. In other words,

$$\mathfrak{cl}(W, \psi, F) = W^\otimes / J_F,$$

in a natural manner. We have then

$$\lambda_F^{-1}[\ker(A)] = J_F \quad (W^\otimes, \circ)/\ker(A) = \mathbf{cl}(W, \psi, F).$$

3. BASIC PROPERTIES OF BRAIDED QUANTUM GROUPS

Let us consider a complex associative algebra \mathcal{A} , with the unit element $1 \in \mathcal{A}$ and the product map $m: \mathcal{A} \otimes \mathcal{A} \rightarrow \mathcal{A}$. Further, let us assume that \mathcal{A} is equipped with a coassociative coalgebra structure, specified by the coproduct $\phi: \mathcal{A} \rightarrow \mathcal{A} \otimes \mathcal{A}$ and the counit $\epsilon: \mathcal{A} \rightarrow \mathbb{C}$. Finally, let us assume that there exist bijective linear maps $\kappa: \mathcal{A} \rightarrow \mathcal{A}$ and $\sigma: \mathcal{A} \otimes \mathcal{A} \rightarrow \mathcal{A} \otimes \mathcal{A}$ such that the following equalities hold:

$$\begin{aligned} (6) \quad & \sigma(m \otimes \text{id}) = (\text{id} \otimes m)(\sigma \otimes \text{id})(\text{id} \otimes \sigma) \\ (7) \quad & \sigma(\text{id} \otimes m) = (m \otimes \text{id})(\text{id} \otimes \sigma)(\sigma \otimes \text{id}) \\ (8) \quad & 1\epsilon = m(\text{id} \otimes \kappa)\phi = m(\kappa \otimes \text{id})\phi \\ (9) \quad & \phi m = (m \otimes m)(\text{id} \otimes \sigma \otimes \text{id})(\phi \otimes \phi) \\ (10) \quad & (\sigma \otimes \text{id}^2)(\text{id} \otimes \phi \otimes \text{id})(\sigma^{-1} \otimes \text{id})(\text{id} \otimes \phi) \\ & = (\text{id}^2 \otimes \sigma)(\text{id} \otimes \phi \otimes \text{id})(\text{id} \otimes \sigma^{-1})(\phi \otimes \text{id}). \end{aligned}$$

By definition [3], every pair $G = (\mathcal{A}, \{\phi, \epsilon, \kappa, \sigma\})$ satisfying the above requirements is called a *braided quantum group*.

The map σ plays the role of the “twisting operator” (the ordinary transposition in the standard theory of Hopf algebras). This operator induces a product in $\mathcal{A} \otimes \mathcal{A}$, via the formula

$$(a \otimes b)(c \otimes d) = a\sigma(b \otimes c)d.$$

Identities (6)–(7) ensure that this defines an associative algebra structure on $\mathcal{A} \otimes \mathcal{A}$, such that $1 \otimes 1$ is the unit element. In particular,

$$\sigma(1 \otimes a) = a \otimes 1 \quad \sigma(a \otimes 1) = 1 \otimes a.$$

for each $a \in \mathcal{A}$. Equality (9) then says that ϕ is multiplicative. Identity (10) expresses the coassociativity of the map $(\text{id} \otimes \sigma^{-1} \otimes \text{id})(\phi \otimes \phi)$.

The antipode κ is uniquely determined by (8). The flip-over operator σ is expressible through ϕ, m and κ in the following way

$$\sigma = (m \otimes m)(\kappa \otimes \phi m \otimes \kappa)(\phi \otimes \phi).$$

Besides the flip-over operator σ , a “secondary” flip-over operator τ naturally enters the game. This map is specified by

$$\tau\sigma^{-1} = (\text{id}^2 \otimes \epsilon)(\text{id} \otimes \sigma^{-1})(\phi \otimes \text{id}) = (\epsilon \otimes \text{id}^2)(\sigma^{-1} \otimes \text{id})(\text{id} \otimes \phi).$$

Its inverse is given by

$$\tau^{-1}\sigma = (\text{id}^2 \otimes \epsilon)(\text{id} \otimes \sigma)(\phi \otimes \text{id}) = (\epsilon \otimes \text{id}^2)(\sigma \otimes \text{id})(\text{id} \otimes \phi).$$

Two maps satisfy the following system of braid equations

$$\begin{aligned}
(\sigma \otimes \text{id})(\text{id} \otimes \sigma)(\sigma \otimes \text{id}) &= (\text{id} \otimes \sigma)(\sigma \otimes \text{id})(\text{id} \otimes \sigma) \\
(\tau \otimes \text{id})(\text{id} \otimes \sigma)(\sigma \otimes \text{id}) &= (\text{id} \otimes \sigma)(\sigma \otimes \text{id})(\text{id} \otimes \tau) \\
(\sigma \otimes \text{id})(\text{id} \otimes \tau)(\sigma \otimes \text{id}) &= (\text{id} \otimes \sigma)(\tau \otimes \text{id})(\text{id} \otimes \sigma) \\
(\sigma \otimes \text{id})(\text{id} \otimes \sigma)(\tau \otimes \text{id}) &= (\text{id} \otimes \tau)(\sigma \otimes \text{id})(\text{id} \otimes \sigma) \\
(\tau \otimes \text{id})(\text{id} \otimes \tau)(\sigma \otimes \text{id}) &= (\text{id} \otimes \sigma)(\tau \otimes \text{id})(\text{id} \otimes \tau) \\
(\tau \otimes \text{id})(\text{id} \otimes \sigma)(\tau \otimes \text{id}) &= (\text{id} \otimes \tau)(\sigma \otimes \text{id})(\text{id} \otimes \tau) \\
(\sigma \otimes \text{id})(\text{id} \otimes \tau)(\tau \otimes \text{id}) &= (\text{id} \otimes \tau)(\tau \otimes \text{id})(\text{id} \otimes \sigma) \\
(\tau \otimes \text{id})(\text{id} \otimes \tau)(\tau \otimes \text{id}) &= (\text{id} \otimes \tau)(\tau \otimes \text{id})(\text{id} \otimes \tau).
\end{aligned}$$

Operators σ and τ naturally generate a “braid system” $\mathcal{T} = \{\sigma_n \mid n \in \mathbb{Z}\}$ consisting of braidings of the form

$$\sigma_n = \tau(\sigma^{-1}\tau)^{-n} = (\tau\sigma^{-1})^{-n}\tau.$$

The following twisting properties hold

$$\begin{aligned}
(\phi \otimes \text{id})\sigma_{n+k} &= (\text{id} \otimes \sigma_k)(\sigma_n \otimes \text{id})(\text{id} \otimes \phi) \\
(\text{id} \otimes \phi)\sigma_{n+k} &= (\sigma_k \otimes \text{id})(\text{id} \otimes \sigma_n)(\phi \otimes \text{id}) \\
\sigma_n(\kappa \otimes \text{id}) &= (\text{id} \otimes \kappa)\sigma_{-n} \\
\sigma_n(\text{id} \otimes \kappa) &= (\kappa \otimes \text{id})\sigma_{-n} \\
\sigma_n(m \otimes \text{id}) &= (\text{id} \otimes m)(\sigma_n \otimes \text{id})(\text{id} \otimes \sigma_n) \\
\sigma_n(\text{id} \otimes m) &= (m \otimes \text{id})(\text{id} \otimes \sigma_n)(\sigma_n \otimes \text{id}).
\end{aligned}$$

The formula

$$\epsilon m = (\epsilon \otimes \epsilon)\sigma^{-1}\tau$$

generalizes the standard multiplicativity property for the counit. A braided counterpart of the standard anti(co)multiplicativity property of the antipode is given by

$$\begin{aligned}
\phi\kappa &= \sigma(\kappa \otimes \kappa)\phi \\
\kappa m &= m(\kappa \otimes \kappa)\tau\sigma^{-1}\tau\sigma^{-1}\tau.
\end{aligned}$$

If $\sigma = \sigma^{-1}$ then all maps σ_n are involutive, too.

4. CONSTRUCTION OF GROUP STRUCTURES

This section is devoted to the construction of the canonical braided quantum group structure on braided Clifford algebras. We shall assume that the braid operator ψ is *involutive*. Let $\mathcal{A}_F = \mathfrak{cl}(W, \psi, F)$ be the corresponding Clifford algebra.

The ideal $\ker(A)$ is generated by the space $\ker(\text{id}^2 - \psi) = \text{im}(\text{id}^2 + \psi)$ consisting of ψ -invariant elements. It follows that the corresponding Clifford ideal J_F is generated by elements of the form

$$(11) \quad Q = \sum_k \{x_k \otimes y_k - F(x_k, y_k)1\}$$

where $\sum_k \psi(x_k \otimes y_k) = \sum_k x_k \otimes y_k$.

We shall denote by m_F the Clifford product in \mathcal{A}_F . Without a lack of generality we can assume that F is ψ -symmetric.

Lemma 1. *There exists the unique linear operator $\sigma_F: \mathcal{A}_F \otimes \mathcal{A}_F \rightarrow \mathcal{A}_F \otimes \mathcal{A}_F$ satisfying*

$$(12) \quad (m_F \otimes \text{id})(\text{id} \otimes \sigma_F)(\sigma_F \otimes \text{id}) = \sigma_F(\text{id} \otimes m_F)$$

$$(13) \quad (\text{id} \otimes m_F)(\sigma_F \otimes \text{id})(\text{id} \otimes \sigma_F) = \sigma_F(m_F \otimes \text{id})$$

$$(14) \quad \sigma_F(x \otimes y) = -\psi(x \otimes y) - F(x, y)1 \otimes 1,$$

for each $x, y \in W$. Furthermore, σ_F is involutive and satisfies the braid equation.

Proof. Equalities of the same form uniquely and consistently determine a linear operator $\sigma_F: W^\otimes \otimes W^\otimes \rightarrow W^\otimes \otimes W^\otimes$. The fact that σ_F is a braid operator follows from the braid equation for ψ , and the covariance property (4). Involutivity of ψ and ψ -symmetricity of F imply that σ_F is involutive. Using the definition of J_F , and (4), as well as the braid equation for ψ it follows that

$$\sigma_F(J_F \otimes W^\otimes) = W^\otimes \otimes J_F$$

$$\sigma_F(W^\otimes \otimes J_F) = J_F \otimes W^\otimes.$$

Hence, σ_F can be projected on $\mathcal{A}_F \otimes \mathcal{A}_F$. \square

In what follows it will be assumed that $\mathcal{A}_F \otimes \mathcal{A}_F$ is endowed with the σ_F -induced product.

Lemma 2. *There exists the unique (unital) homomorphism $\phi: \mathcal{A}_F \rightarrow \mathcal{A}_F \otimes \mathcal{A}_F$ satisfying*

$$(15) \quad \phi(x) = 1 \otimes x + x \otimes 1$$

for each $x \in W$.

Proof. There exists the unique unital homomorphism $\phi: W^\otimes \rightarrow \mathcal{A}_F \otimes \mathcal{A}_F$ satisfying the above requirement. We have

$$\phi(J_F) = \{0\}.$$

Indeed it is sufficient to check that ϕ vanishes on elements of the form (11). A direct calculation gives

$$\begin{aligned} \phi(Q) &= \sum_k \left\{ x_k y_k \otimes 1 + x_k \otimes y_k + \sigma_F(x_k \otimes y_k) + 1 \otimes x_k y_k - F(x_k, y_k)1 \otimes 1 \right\} \\ &= \sum_k \left\{ (x_k y_k - F(x_k, y_k)1) \otimes 1 + 1 \otimes (x_k y_k - F(x_k, y_k)1) \right\} = 0. \end{aligned}$$

Hence ϕ can be factorized through the ideal J_F . In such a way we obtain the desired map $\phi: \mathcal{A}_F \rightarrow \mathcal{A}_F \otimes \mathcal{A}_F$. \square

Let $\tau: \mathcal{A}_F \otimes \mathcal{A}_F \rightarrow \mathcal{A}_F \otimes \mathcal{A}_F$ be a (involutive) brading given by

$$\tau(a \otimes b) = (-1)^{\partial a \partial b} \psi(a \otimes b)$$

where the grading is induced from W^\wedge . We have

$$(16) \quad \tau(m_F \otimes \text{id}) = (\text{id} \otimes m_F)(\tau \otimes \text{id})(\text{id} \otimes \tau)$$

$$(17) \quad \tau(\text{id} \otimes m_F) = (m_F \otimes \text{id})(\text{id} \otimes \tau)(\tau \otimes \text{id})$$

as directly follows from (2)–(3) and (4)–(5).

Lemma 3. *Operators τ and σ_F are mutually compatible, in the braided sense.*

Proof. Using (14) and (4) we find that the “braided compatibility” between τ and σ_F holds on the space $W \otimes W \otimes W$. Applying inductively (12)–(13) and (16)–(17) we conclude that the braided compatibility is preserved if we pass to the higher filtrant spaces. \square

Operators τ and σ_F express twisting properties of the coproduct map

Lemma 4. *The following identities hold*

$$\begin{aligned} (\phi \otimes \text{id})\sigma_F &= (\text{id} \otimes \tau)(\sigma_F \otimes \text{id})(\text{id} \otimes \phi) = (\text{id} \otimes \sigma_F)(\tau \otimes \text{id})(\text{id} \otimes \phi) \\ (\phi \otimes \text{id})\tau &= (\text{id} \otimes \tau)(\tau \otimes \text{id})(\text{id} \otimes \phi) \\ (\text{id} \otimes \phi)\sigma_F &= (\tau \otimes \text{id})(\text{id} \otimes \sigma_F)(\phi \otimes \text{id}) = (\sigma_F \otimes \text{id})(\text{id} \otimes \tau)(\phi \otimes \text{id}) \\ (\text{id} \otimes \phi)\tau &= (\tau \otimes \text{id})(\text{id} \otimes \tau)(\phi \otimes \text{id}). \end{aligned}$$

Proof. All these equalities trivially hold on $W \otimes W$. Inductively applying (16)–(17) and (12)–(13), the braided compatibility between τ and σ_F and the multiplicativity of ϕ we conclude that the above equalities hold on the whole $\mathcal{A}_F \otimes \mathcal{A}_F$. \square

As a simple consequence we find

Lemma 5. *The following identity holds*

$$(18) \quad (\phi \otimes \text{id})\phi = (\text{id} \otimes \phi)\phi.$$

Proof. Let us consider the space $\mathcal{A}_F \otimes \mathcal{A}_F \otimes \mathcal{A}_F$, endowed with the product M specified by

$$M \leftrightarrow (m_F \otimes m_F \otimes m_F)(\text{id} \otimes \sigma_F \otimes \sigma_F \otimes \text{id})(\text{id}^2 \otimes \tau \otimes \text{id}^2).$$

Twisting properties of ϕ (and the multiplicativity) imply that both sides of (18) are unital homomorphisms. On the other hand they trivially coincide on the generating space W . Hence (18) holds on the whole \mathcal{A}_F . \square

To complete the construction let us introduce the counit and the antipode.

Lemma 6. (i) *There exists the unique linear map $\epsilon: \mathcal{A}_F \rightarrow \mathbb{C}$ satisfying*

$$(19) \quad \epsilon m_F = (\epsilon \otimes \epsilon) \sigma_F \tau$$

$$(20) \quad \epsilon(W) = \{0\} \quad \epsilon(1) = 1.$$

(ii) *There exists the unique linear map $\kappa_F: \mathcal{A}_F \rightarrow \mathcal{A}_F$ such that*

$$(21) \quad \kappa_F m_F = m_F (\kappa_F \otimes \kappa_F) \tau \sigma_F \tau \sigma_F \tau$$

$$(22) \quad \kappa_F(1) = 1 \quad \kappa_F(x) = -x$$

for each $x \in W$.

(iii) *We have*

$$(23) \quad m_F(\kappa_F \otimes \text{id})\phi = m_F(\text{id} \otimes \kappa_F)\phi = 1\epsilon$$

$$(24) \quad (\epsilon \otimes \text{id})\phi = (\text{id} \otimes \epsilon)\phi = \text{id}.$$

Proof. Formulas of the same form as (19)–(22) consistently and uniquely determine tensor algebra maps $\epsilon: W^\otimes \rightarrow \mathbb{C}$ and $\kappa_F: W^\otimes \rightarrow W^\otimes$. The consistency follows from the braided compatibility of $\{\sigma_F, \tau\}$ (understood as braidings on W^\otimes). It is therefore sufficient to prove that $\epsilon(J_F) = \{0\}$ and $\kappa_F(J_F) = J_F$. In view of the braided-covariance of the generator relations space J_F^2 it is sufficient to check that $\epsilon(J_F^2) = \{0\}$ and $\kappa_F(J_F^2) = J_F^2$. A direct computation gives

$$\epsilon(Q) = 0 \quad \kappa_F(Q) = -Q.$$

Hence, maps ϵ and κ_F are factorizable through J_F . From the definition of ϵ it follows that

$$(\epsilon \otimes \text{id})\tau = \text{id} \otimes \epsilon$$

$$(\text{id} \otimes \epsilon)\tau = \epsilon \otimes \text{id}.$$

Finally, let us check the antipode and the counit axioms. It is evident that these equalities hold on the space W . On the other hand, from (19), (21), the multiplicativity of ϕ and the twisting properties of ϕ and ϵ it follows that elements on which equalities (23)–(24) hold form a subalgebra of \mathcal{A}_F . Hence (23)–(24) hold on the whole \mathcal{A}_F . \square

Combining all derived properties we conclude that $G = (\mathcal{A}_F, \{\phi, \epsilon, \kappa_F, \sigma_F\})$ is a braided quantum group.

The presented construction can be performed in a conceptually different way, explicitly working in terms of the exterior algebra W^\wedge .

Let us consider the space $W \oplus W$, endowed with a braid operator Ψ , given by the block matrix

$$\Psi = \begin{pmatrix} \psi & 0 & 0 & 0 \\ 0 & 0 & \psi & 0 \\ 0 & \psi & 0 & 0 \\ 0 & 0 & 0 & \psi \end{pmatrix}$$

and with the quadratic form E defined as follows. Its restriction on “homogeneous” summands in the tensor square of $W \otimes W$ coincides with F , while $E = -F/2$ on “mixed” terms. Let $\Delta: W \rightarrow W \oplus W$ be the standard diagonal map.

Lemma 7. *There exists the unique algebra map $\Delta^*: \mathcal{A}_F \rightarrow \mathfrak{cl}(W \oplus W, \Psi, E)$ extending Δ . The map Δ^* is a homomorphism of the corresponding exterior algebras, too.*

Proof. By definition, Δ intertwines braids ψ and Ψ . This implies that Δ is uniquely extendible to a unital homomorphism of the form $\Delta^*: W^\wedge \rightarrow (W \oplus W)^\wedge$. The same map is also a homomorphism of associated Clifford algebras, as follows from the equality

$$E(\Delta(x), \Delta(y)) = F(x, y)$$

and expression (5) for the Clifford product. \square

Algebra \mathcal{A}_F can be embedded in $\mathfrak{cl}(W \otimes W, \Psi, E)$ in two different natural ways, with the help of monomorphisms $\ell_\pm: \mathcal{A}_F \rightarrow \mathfrak{cl}(W \otimes W, \Psi, E)$ specified by

$$\ell_-(x) = (x, 0) \quad \ell_+(x) = (0, x)$$

where $x \in W$. The maps ℓ_\pm are also embeddings of corresponding exterior algebras. The formula

$$\ell_F(a \otimes b) = \ell_-(a)\ell_+(b)$$

defines a linear map $\ell_F: \mathcal{A}_F \otimes \mathcal{A}_F \rightarrow \mathfrak{cl}(W \otimes W, \Psi, E)$.

Lemma 8. *The map ℓ_F is an algebra isomorphism.*

Proof. It is sufficient to observe that

$$\ell_+(x)\ell_-(y) = - \sum_k \ell_-(y_k)\ell_+(x_k) - F(x, y)1$$

where $x, y \in W$ and $\sum_k y_k \otimes x_k = \psi(x \otimes y)$. This gives all mutual relations between elements of two embedded subalgebras. \square

We have

$$(25) \quad \Delta^* = \ell_F \phi.$$

Indeed, both sides of (25) are unital homomorphisms trivially coinciding on W , by construction.

5. CONCLUDING REMARKS

Let us analyze a special class of braided Clifford algebras, admitting spinor representations of a classical type. Geometrically these algebras represent particularly regular quantum spaces, possessing “sufficiently large” symmetry groups. Let us assume that F is nondegenerate, and that

$$W = W_- \oplus W_+$$

where W_\pm are F -isotropic subspaces (mutually naturally dual). Further, let us assume that the above decomposition is compatible with the braid ψ , in the sense that

$$\psi(W_i \otimes W_j) = W_j \otimes W_i$$

for $i, j \in \{+, -\}$.

The corresponding exterior algebras W_{\pm}^{\wedge} are understandable as subalgebras of \mathcal{A}_F , in a natural manner. Moreover, the map $\mu_F: W_{-}^{\wedge} \otimes W_{+}^{\wedge} \rightarrow \mathcal{A}_F$ given by

$$\mu_F(a \otimes b) = ab$$

is bijective. The corresponding “spinor space” can be defined as follows [2]. The restriction $\epsilon_{+} = \epsilon|_{W_{+}^{\wedge}}$ is multiplicative. In particular, ϵ_{+} gives a left W_{+}^{\wedge} -module structure on \mathbb{C} . On the other hand, \mathcal{A}_F is a right W_{+}^{\wedge} -module, in a natural manner. Let \mathcal{S} be a left \mathcal{A}_F -module, given by

$$\mathcal{S} = \mathcal{A}_F \otimes_{*} \mathbb{C}$$

where the tensor product is taken over W_{+}^{\wedge} .

The “spinor module” \mathcal{S} is simple, and the action of \mathcal{A}_F is faithful. In particular, if \mathcal{A}_F is finite-dimensional (this is a property of the initial braid ψ) then a natural isomorphism $\mathcal{A}_F \leftrightarrow \text{lin}(\mathcal{S})$ holds. The corresponding quantum space is completely homogeneous and pointless.

Classical Clifford and Weyl algebras are included in the formalism in a trivial way.

According to Lemma 7 the map Δ^{*} is F -independent. It turns out that ϕ , and hence ϵ , are F -independent, too.

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INSTITUTO DE MATEMATICAS, UNAM, AREA DE LA INVESTIGACION CIENTIFICA, CIRCUITO EXTERIOR, CIUDAD UNIVERSITARIA, MÉXICO DF, CP 04510, MEXICO

Written In

FACULTAD DE ESTUDIOS SUPERIORES CUAUTITLAN, UNAM, MEXICO